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# Trouble with $\xi$ scaling?*

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We review our analysis of inclusive lepton-hadron scattering, which is based on the use of the variable $\xi$ that incorporates target- and constituent-mass effects in a color-gauge theory [quantum chromodynamics (QCD)]. We show that the apparent paradoxes recently encountered by several groups in connection with $\xi$ scaling are not paradoxes at all, but rather are based on misunderstanding of our analysis. Paradoxes arise when the $\xi$-scaling analysis is applied for all $\xi$, but, as we stated in earlier work, QCD predicts that corrections (due to twist-greater-than-2 operators) are negligible only for $W$ well above the proton mass. We restate our previous analysis in which we introduced a smooth function $F_s$ that $\xi$-scales up to explicit logarithmic corrections. $F_s$ has no thresholds (it is nonzero for $0 < \xi < 1$). QCD predicts that $F_s$ describes the data in the sense of Bloom-Gilman local duality. More constructively, we use the parton-model language to interpret the field-theoretic operator-product expansion in successive "twists" and develop an intuitive physical interpretation of twist. We reanalyze the "paradoxes" in parton language. We resolve (to our satisfaction) the question of nonperturbative effects in the analysis of electroproduction.

## I. INTRODUCTION

The concept of Bjorken scaling has had considerable success in the analysis of deep-inelastic electroproduction data. Theoretical attempts to modify the scaling laws or the scaling variable to include the kinematic effects of target and constituent masses have met with some skepticism in the past. The controversy over mass-dependent scaling variables has resurfaced after our recent analysis of inclusive lepton-hadron scattering in the framework of a color-gauge theory of the strong interactions[1,2] (quantum chromodynamics: QCD). To clear up the confusion, we wish to present the issues and arguments simply, stressing physical principles and minimizing inessential technicalities.

In Ref. 1 (henceforth GP) we showed how the kinematic effects of target and constituent masses can be summarized by an appropriate scaling variable $\xi$. For example, $\xi$ describes the nature of the threshold for production of states containing heavy quarks in inclusive neutrino scattering (i.e., $\nu + p \rightarrow \mu^- +$ any state containing a heavy quark). In Ref. 2 (which we will call DGP), the logical structure of our analysis (only implicit in GP) is discussed in detail. We incorporate the leading interaction corrections in color-gauge theory and obtain an accurate description of the scaling violations observed in electroproduction at SLAC, as well as a quantitative theoretical understanding of local duality. The question is whether these successes are fortuitous and, more generally, whether color dynamics is a predictive theory at currently accessible energies.

These issues have been raised recently by several groups.[3-5] We begin with a discussion of the criticisms of our work. At this point, a reader who is uncomfortable with the renormalization group and the operator-product expansion (OPE) may worry that he is about to be subjected to a hair-splitting argument over some mathematical arcanum. This is not so. The points at issue in the recent controversies are questions of logic, arising from a misunderstanding of our approach. Consequently we attempt to present the arguments clearly and concisely, in a form that should be easily accessible to any reader who has gone this far. The criticisms are sharpest in the work of Gross, Treiman, and Wilczek[5] (GTW), who express the general misgivings of all the critics in the form of paradoxes. They prove a number of mathematical facts and draw apparently paradoxical conclusions. We hope to convince the patient reader that there are no paradoxes.

For the sake of the impatient and the "do-it-yourself" reader, we first present the plan of the paper in some detail. In Sec. II we review the GTW paradox and our resolution in the moment language appropriate to the OPE. Essentially, the paradox arises from the fact that the physical thresholds occur at fixed values of $W^2$, not at fixed $\xi$. A function of $\xi$ which describes a structure function at some $Q^2$ cannot describe it accurately at higher $Q^2$ because the thresholds are in the wrong place.[4] More formally, if the $\xi^n$ moments of a structure function are equal *for all n* at two different $Q^2$ (up to the logarithmic corrections of asymptotic freedom), GTW prove that the structure function at the higher $Q^2$ is zero in a region of $\xi$ where no physical constraint forces it to vanish. The resolution is that for the large-$n$ values necessary to derive the paradox, the underlying color-gauge theory *predicts* that the $\xi^n$ moments are not even approximately equal.

In the next two sections, we restate our resolution in two different languages. In Sec. III we make the logical connection between our analysis (of the contribution of twist-2 operators to the structure functions) and the "scaling-limit function" of earlier works. We determine the $Q^2$ dependence of a *smooth* function $F_s(\xi, Q^2)$ which is nonzero in the entire $\xi$ interval 0 to 1. It contains no threshold information, and is the analog in our language of the scaling-limit curve. $F_s$ is a good approximation to the structure function at small $\xi$, but as $\xi$ increases into the resonance region, the oscillations of the physical function around $F_s$ become more pronounced. Even in the resonance region $F_s$ still describes the structure function in the sense of local duality.[6]

In Sec. IV, an analogous paradox is presented in the language of the parton model. One uses the impulse approximation to describe the fundamental scattering process, and the struck parton is treated as a free particle. Does consistency require that the probability distribution of partons also be given by free field theory? The answer is no: Binding effects as represented by a $\langle p_T^2\rangle \neq 0$ are negligible in the photon-parton differential cross section for large $Q^2$, while the initial parton distribution will be nontrivial as long as $\langle p_T^2\rangle / m_{\text{proton}}{}^2 \neq 0$. In Sec. IV we go on to discuss the physics of the OPE twist analysis in terms of initial and final state interactions.

Section V is devoted to answering a variety of questions that surface as afterthoughts once local duality is understood. We discuss the possibility of "nonanalytic" effects, undetectable in perturbation theory, the nonparadox of $\xi$ scaling and truly free fields, the peculiarities of heavy quarks, i.e., $m_{\text{quark}} > m_{\text{proton}}$, and the difficulties in going to small $\xi$.

## II. IS THERE A PARADOX?

### The paradox, ignoring interactions

The traditional Bjorken scaling variable is $x = Q^2/2p\cdot q = Q^2/2m\nu$, where $m$ is the target mass. The invariant mass squared of the hadronic debris in the final state is $W^2 = m^2 + 2m\nu - Q^2$ (we denote $Q^2 = -q^2 > 0$). The variable $\xi$ is

$$\xi = \frac{2x}{1+(1+4m^2x^2/Q^2)^{1/2}} \tag{1a}$$

$$= \frac{1}{2}\left\{1 + \frac{W^2-m^2}{Q^2} + \left[1 + 2\,\frac{W^2+m^2}{Q^2} + \left(\frac{W^2-m^2}{Q^2}\right)^2\right]^{1/2}\right\}^{-1}, \tag{1b}$$

where we have set the quark masses equal to zero because they are irrelevant to the logical issues.

We assert that $\xi$ is the right variable to use in an analysis that takes into account the $m^2/Q^2$ corrections to Bjorken scaling. If we ignore the color-gauge interactions, the result is very simple. For a suitably chosen structure function $F(\xi, Q^2)$, then $n \geq 2$ moments are independent of $Q^2$:

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = a_n\,. \tag{2}$$

For reasonable functions $F$, Eq. (2) implies that $F(\xi, Q^2)$ is $Q^2$-independent; it scales in $\xi$. Thus if the structure function has thresholds, they must be fixed-$\xi$ thresholds. But the physical structure functions have fixed-$W$ thresholds and a $W=m$ pole. These physical singularities move in $\xi$ as $Q^2$ changes, as one can read off Eq. (1b). *Exact* $\xi$ scaling cannot reproduce the physical thresholds. This is the essence of the paradox, as stated clearly in Ref. 4. To underline the logical issues to be discussed below, we proceed to restate the paradox in the language of GTW.

Let $\xi_{\max}(Q^2)$ be the position in $\xi$ of the elastic-scattering pole ($x=1$):

$$\xi_{\max}(Q^2) = 2/[1+(1+4m^2/Q^2)^{1/2}]\,. \tag{3}$$

Energy-momentum conservation implies that $F(\xi, Q^2)$ vanishes for $\xi > \xi_{\max}(Q^2)$. Now use Eq. (2) to relate the structure function at $Q^2$ to the structure function at $Q_0^2 > Q^2$:

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = \int_0^1 d\xi\, \xi^n F(\xi, Q_0^2)\,. \tag{4}$$

Since $F(\xi, Q^2)$ vanishes for $\xi > \xi_{\max}(Q^2)$, Eq. (4) implies that $F(\xi, Q_0^2)$ (which in this simple case is just the same function of $\xi$) must also vanish for $\xi > \xi_{\max}(Q^2)$. But $F(\xi, Q_0^2)$ is physically constrained to vanish only for $\xi > \xi_{\max}(Q_0^2)$. The paradox is simply that the moment statements Eq. (2), imply that $F(\xi, Q_0^2)$ vanishes for $\xi_{\max}(Q_0^2) > \xi > \xi_{\max}(Q^2)$, where there is no physical reason for it to vanish. Logically, one must choose between the following two conclusions:

1. The $\xi$ variable is useless, and there is no hope of including $m^2/Q^2$ corrections in a field-theoretic treatment of inclusive lepton-hadron scattering.
2. The color-gauge interactions cannot be ignored.

The second alternative sounds rather reasonable, and suggests that one should look for a solution to the paradox in the interaction corrections to Eq. (2). In QCD, the most important interaction corrections at large $Q^2$ arise from the anomalous dimensions and nontrivial coefficient functions of the twist-2 operators. These modify Eq. (2) to give the moments a $\log Q^2$ dependence:

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = A_n(Q^2) = a_n\left[\ln\left(\frac{Q^2}{\Lambda^2}\right)\right]^{-d_n}\left[1+O\left(\frac{\ln n}{\ln(Q^2/\Lambda^2)}\right)\right] \quad (5)$$

with $\Lambda$ a constant and $d_n > 0$ (we ignore the complication of operator mixing which is irrelevant to this argument). As GTW correctly prove, these interaction corrections do not defuse the paradox. Equation (5) still implies that $F(\xi, Q_0^2)$ vanishes where it should not. The resolution of the paradox lies deeper.

### There is no paradox

In DGP we argue that the logarithmic corrections of Eq. (5) are not the only ones that are relevant in practice. Further corrections modify Eq. (5) to read

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = A_n(Q^2) + \sum_{k=1}^{\infty}\left(n\,\frac{M_0^2}{Q^2}\right)^k B_{n,k}(Q^2)\,. \quad (6)$$

Field-theory experts will recognize Eq. (6) as a realization of the classic twist analysis of Gross and Treiman,[7] where the $k$th term in the sum is the contribution of operators with twist $2k+2$. The ratio $B_{n,k}(Q^2)/A(Q^2)$ has no power dependence on $n$, $k$, or $Q^2$; so we fix the mass parameter $M_0$ by defining this ratio to be of order 1. In DGP we argue that $M_0$ should be of the order of a few hundred MeV. We then use SLAC data[8] from both the scaling region and the resonance region to determine $M_0$ experimentally by isolating the $k=1$ contribution to Eq. (6). The DGP result is $M_0 = (375 \pm 25)$ MeV, in agreement with expectations.

It is the second term on the right-hand side of Eq. (6) that tames the paradox. To see this, let us estimate the values of $n$ used in deriving the paradox. For large $\xi$, the measured structure functions behave as $(1-\xi)^a$ with $a = 3$ to 4. The function $\xi^n(1-\xi)^a$ has a maximum at

$$\xi_n = \frac{n}{n+a} \simeq 1 - \frac{a}{n} \quad (n \text{ large})\,, \quad (7)$$

so that the $n$th moment is sensitive to the behavior of the structure function for $\xi \simeq \xi_n$. To obtain the paradoxical conclusion, the GTW analysis must be sensitive to $\xi$ near the threshold region:

$$\xi \simeq \xi_{\max}(Q^2) = 1 - \frac{m^2}{Q^2} + O\left(\frac{m^4}{Q^4}\right)\,. \quad (8)$$

GTW must take $n$ large enough so that $\xi_n \simeq \xi_{\max}(Q^2)$, or

$$n \simeq \frac{aQ^2}{m^2}\,. \quad (9)$$

But for such large $n$, the second term on the right-hand side of Eq. (6) cannot be neglected. Indeed, since

$$n\frac{M_0^2}{Q^2} \simeq \frac{aQ^2}{m^2}\,\frac{M_0^2}{Q^2} \simeq 1\,,$$

all $k$ in the sum contribute. Since these terms cannot be ignored for $n$ sensitive to the threshold behavior, *there is no paradox.*

The resolution may be summarized succinctly by noting that there is a nonuniformity in the limits $n \to \infty$ (or $\xi \to 1$) and $Q^2 \to \infty$. The appearance of powers of $nM^2/Q^2$ in Eq. (6) signals a breakdown of the theoretical techniques at small $W$, e.g. $1 \lesssim W \lesssim 2$ GeV. Such nonuniformity was first studied in detail by Gross[9] in terms of $\log n/\log(Q^2/\Lambda^2)$ terms that appear in Eq. (5). Their origin and consequences are essentially the same as the $n/Q^2$ terms in Eq. (6), but over the range $1 \leq Q^2 \leq 15$ GeV$^2$ and $W \gtrsim 1$ GeV they are less important.

## III. IS $\xi$ SCALING USEFUL?

We hope the reader is convinced that the $\xi$-scaling analysis does not lead to paradoxes, but he may still wonder whether it is any real improvement over $x$ scaling. Are the higher-twist contributions to Eq. (6) large enough to render the whole approach useless? The answer is no. Because $M_0^2$ is much smaller than $m^2$ (by almost an order of magnitude), there are interesting regions where the differences between $\xi$ and $x$ is important and where $\xi$ scaling is still useful. To show this in detail, we review briefly how DGP makes contact with experimental data.

We define a function $F_s(\xi, Q^2)$, which is smooth and nonzero in the entire region $0 < \xi < 1$, and whose moments are the $A_n(Q^2)$ of Eq. (5) and Eq. (6):

$$\int_0^1 d\xi\, \xi^n F_s(\xi, Q^2) = A_n(Q^2)\,. \quad (10)$$

The $Q^2$ dependence of $F_s$ can be calculated perturbatively (if we ignore operator mixing; it depends only on the constant $\Lambda^2$ and the value of $F_s$ at some fixed $Q^2 = Q_0^2$).

The smooth function $F_s$ is related to the physical structure function $F(\xi, Q^2)$ by Eq. (6). The relation is qualitatively different in the scaling region, the resonance region, and the threshold region, which we now discuss in turn. To guide the eye, we refer to Fig. 1, reproduced from DGP. In this figure, the smooth dashed curve is the contribution to $\nu W_2$ from $F_s(\xi, Q^2 = 2\text{ GeV}^2)$, and the

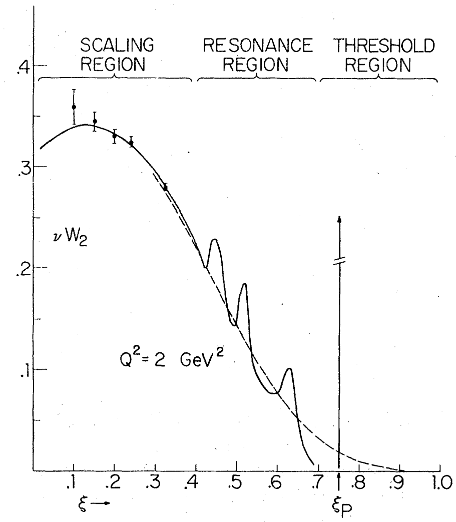


FIG. 1. An illustration of local duality from DGP. The dotted curve is the contribution of twist-2 operators to $\nu W_2$ at $Q^2 = 2$ GeV$^2$. The solid curve is an experimental fit to $\nu W_2$ including the resonance region (assuming that $R = 0.18$). The data points are SLAC data in which $\nu W_2$ and $W_1$ are separated.

bumpy curve is a fit to the experimental $\nu W_2(\xi, Q^2 = 2\ \text{GeV}^2)$, containing the observed resonances.

Consider the region $\xi \ll \xi_{\max}(Q^2) \approx 1 - m^2/Q^2$. Moments are sensitive to this region for $n$ such that $\xi_n \simeq 1 - a/n \ll 1 - m^2/Q^2$, or $n \ll aQ^2/m^2$. For such $n$, the second term on the right-hand side Eq. (6) is negligible, so in this region we expect $F(\xi, Q^2) \simeq F_s(\xi, Q^2)$. This is the scaling region at large $W^2$, where the structure function is smooth. The effect of the $\xi$ variable in this region is not at all negligible. In fact, at SLAC energies, a significant fraction of the scaling violation (in $x$ for $x > 0.25$) arises from the $\xi$-scaling effects; the rest comes from logarithmic interaction corrections. To get a more quantitative picture of the significance of $\xi$, compare $\xi$ to $x$ and $x_s$ (see Fig. 2). $x_s$ is a phenomenological, improved scaling variable which summarizes the scaling violations observed at SLAC. For protons

$$x_s = \frac{x}{1 + x(1.4\ \text{GeV}^2)/Q^2}, \qquad (11)$$

$\xi$ lies between $x$ and $x_s$. So scaling violations, which are effectively absent when the SLAC data are expressed in $x_s$, are much weaker in $\xi$ than in $x$.

The intermediate region, $\xi \lesssim \xi_{\max}(Q^2)$, is the resonance region. It corresponds to $n \lesssim aQ^2/m^2$. For such $n$, the higher-twist contributions are small but not negligible. They cause the physical $F$ to oscillate around $F_s$, which is our translation of local duality. At some $\xi$, the higher-twist effects may add constructively to make $F$ much larger than $F_s$, but then $F$ must fall back below $F_s$ at neighboring $\xi$ because the higher-twist contribution to the relevant moments is not large.

In the threshold region $\xi \gtrsim \xi_{\max}(Q^2)$, the theory is compatible with the vanishing of the physical cross section because all terms in Eq. (6) contribute.

It may be helpful to compare our smooth function $F_s$ to the "scaling-limit curve" of Bloom and Gilman,[6] for they are very similar logical constructs. Bloom and Gilman assumed that $\nu W_2(x, Q^2)$ has a well-defined smooth limit $F(x)$ as $Q^2$ goes to $\infty$. They then found that for moderate $Q^2$, the structure function in the scaling region behaved approximately as

$$\nu W_2(x, Q^2) = F(x'), \qquad (12)$$

where $x'$ is the celebrated Bloom-Gilman variable, $x' = Q^2/(2p \cdot q + m^2)$. For large $x'$, in the resonance region, they noticed that the structure function oscillated about the smooth curve $F(x')$.

Our $F_s(\xi, Q^2)$ is the analog of $F(x')$. It is not simply a function of an improved scaling variable because color-gauge theories predict an explicit pattern of logarithmic scaling violation. But like $F(x'(x, Q^2))$, our $F_s(\xi, Q^2)$ is a function of two variables which is determined by its value at any fixed $Q^2 = Q_0^2$ (ignoring operator mixing).

To determine the low moments of $F_s(\xi, Q^2)$ or, equivalently, its form at low $\xi$, an experimental input at moderate $Q_0^2$ is enough. To determine high moments of $F_s(\xi, Q^2)$, or, equivalently, its form at large $\xi$, an experimental input at higher $Q_0^2$ is necessary.

In other words, the smooth function cannot be determined completely by the bumpy $\nu W_2$ for any finite $Q^2$. In particular, it is wrong to say that

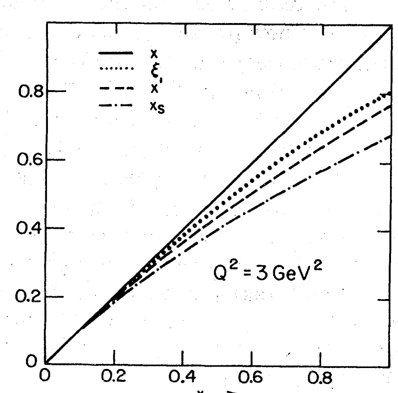


FIG. 2. A comparison of various scaling variables, evaluated at $Q^2 = 3$ GeV$^2$. $x_s$ is a phenomenological variable chosen to summarize the scaling violations observed at SLAC; see Eq. (11).

either approach is incorrect simply because $\nu W_2$ vanishes for $x'$ or $\xi$ close to 1. Rather, the logic proceeds in the opposite direction. Given the smooth function, we know $\nu W_2$ approximately. The approximation is useful only in an average sense in the resonance region, and it improves, becoming locally better, as $x$ decreases into the scaling region. The best we can do in either approach is to construct a smooth function which gives the best fit to $\nu W_2$ in this approximate sense.

The distinction between the resonance region and the scaling region is rather arbitrary. For finite $Q^2$, the structure function is not completely smooth for any $x$. There are always nearby thresholds causing oscillations of $\nu W_2$ about $F_s$. What happens in the scaling region is that these oscillations are so small and closely spaced that they are smeared out by finite experimental resolutions.

It is true that perturbation theory cannot yield detailed information about the bumps in the structure functions. But we can use perturbation theory, appropriately renormalization-group-improved, to study the $Q^2$ dependence of the smooth function $F_s(\xi, Q^2)$. Here the variable $\xi$ is useful because it correctly includes the target- and constituent-mass effects. The $M_0^2/Q^2$ effects of higher-twist operators are smaller by an order of magnitude in the scaling region. Other effects, such as fourth-order corrections to twist-2, are even less important at SLAC energies.

## IV. IN PARTON LANGUAGE

### More paradoxes?

Apparently paradoxical conclusions may also be reached in a parton-model description of our analysis. Scaling in the $\xi$ variable can be obtained in a parton approach that keeps terms of order $m^2/Q^2$. This is done with the usual assumptions (the impulse approximation, on-shell quarks with zero mass), but a new prescription is necessary to modify the statement that $p_\mu(\text{quark}) = xp_\mu$ (target), which is inconsistent if $m_p \neq 0$. The correct field-theoretic result is obtained with the prescription $p^+(\text{quark}) = \xi p^+(\text{target})$, where $p^\pm = (p_0 \pm p_3)$ are "light-cone variables." This leads to $\xi$ scaling, with $\xi = -q^2/p_+ q_-$, in agreement with Eq. (1), provided that effects of order $\langle p_T^2\rangle/Q^2$ are neglected. Here $\vec{p}_T$ is the transverse parton momentum in the collinear photon-target frame. We expect the quantity $\langle p_T^2\rangle$ to be related to the mean square radius of the target, $\langle p_T^2\rangle \approx 3^{2/3}\langle r_p^2\rangle^{-1} = (400\text{ MeV})^2$ for a proton.

Paradoxes arise as follows: Exact $\xi$ scaling is obtained in the approximation $\langle p_T^2\rangle/Q^2 = 0$. If $\langle p_T^2\rangle \equiv 0$, the quarks cannot be confined in the transverse direction. Consequently, they are not con-

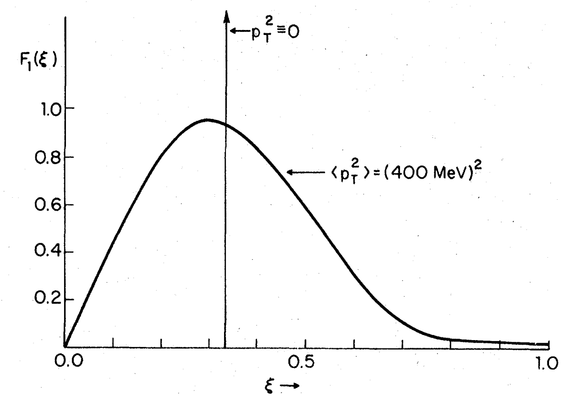


FIG. 3. The limiting distribution $F_1$ of three free quarks in a sphere of radius $R = 1$ F, $\langle p_T^2\rangle = (400\text{ MeV})^2$, based on a bag-model calculation by Jaffe, shows the broadening due to confinement in contrast to the free quark $\delta(\xi - \frac{1}{3})$.

fined in any direction, i.e., they are free. But if the proton is simply a system of three free quarks (at rest in the proton rest frame), then $\nu W_2 \propto \delta(x - \frac{1}{3})$. Hence it is internally inconsistent to assume that $\nu W_2$ is anything but a $\delta$ function once we have assumed that $\langle p_T^2\rangle = 0$. To resolve this paradox we must ask: What are the consequences of a small $\langle p_T^2\rangle \sim (400\text{ MeV})^2$? While $\langle p_T^2\rangle/Q^2$ may be tiny at some large $Q^2$, the fact of confinement as evidenced by $p_T^2 \neq 0$ may drastically alter the appearance of $\nu W_2$, no matter how large $Q^2$ may be. As a qualitative but specific example consider the following "bag" calculation. Jaffe has computed what $\nu W_2$ would look like for a proton made of three noninteracting massless quarks confined to a sphere or radius $R = 1$ F.[10] The $\delta$ function $\nu W_2$ typical of free quarks is broadened by confinement. In Jaffe's approximation, $\nu W_2$ is $Q^2$-independent and has a shape quite like the observed structure function (see Fig. 3). In the same model the momentum per quark is $2.02/R \sim 400$ MeV. Thus there will be a range in $Q^2$ for which $\langle p_T^2\rangle/Q^2$ scaling violation effects are small compared to $m_{\text{proton}}^2/Q^2$. However, the effects of nonvanishing $\langle p_T^2\rangle$ on the quark distribution functions are not small.

### Physics of higher twists

In the OPE analysis, the twist expansion gives a description of $\xi$ moments of structure functions in powers of $M_0^2/Q^2$ [see Eq. (6)] with twist-$2n$ corresponding to the $(M_0^2/Q^2)^{2n-2}$ term. The leading (twist-2) contribution corresponds to the parton-model impulse approximation. The $\log(Q^2/\Lambda^2)$ interaction corrections to twist-2 correspond to vertex corrections and gluon bremsstrahlung in

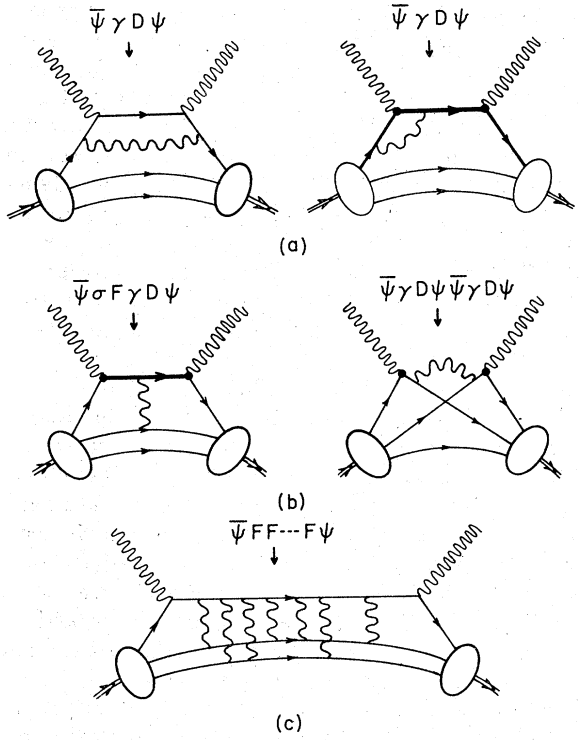


FIG. 4. Diagrams illustrating the meaning of twist in electroproduction with the corresponding operator written above each picture. The solid lines are quark lines coming from a proton. The external wavy lines are photons and the internal wavy lines are colored gluons. (a) shows typical twist-2 effects, corresponding to a parton-model picture with no communication between struck quarks and spectator quarks. (b) shows twist-4 effects. (c) shows a high-twist effect.

the photon–struck-quark interaction [see Fig. 4(a)]. In what follows we develop an intuitive understanding of the phenomena related to twist greater than 2 (see Fig. 4). Particular attention is given to twist-4, which we were able to study phenomenologically in DGP.

There are wave-function (or initial-state) effects and final state effects contained in the terms of twist-4 and higher. To get a feel for the initial-state effects, consider the following parton discussion. The quarks will in general have a $p_T \neq 0$ and be off-shell by some amount proportional to $\langle p_T^2\rangle$. Let their distribution be described by a function $f_i(z, p_T^2)$, where $z$ is the momentum fraction carried by the quark in the + light-cone direction (as in the leading $p_T = 0$ analysis). Repeat the parton-model impulse approximation calculation to get the structure function as an incoherent sum over the constituent partons. Integrate over $z$ and $p_T$ to obtain

$$F \propto \sum_i Q_i^2 \int dz\, dp_T^2 f_i(z, p_T^2)\,\delta(z-\xi_{\mathrm{eff}}) = \int dp_T^2 f(\xi_{\mathrm{eff}}, p_T^2)\,, \tag{13a}$$

where

$$\xi_{\mathrm{eff}} = \xi\left[1+\frac{p_T^2}{Q^2}+O\left(\frac{p_T^4}{Q^4}\right)\right]. \tag{13b}$$

Expand in powers of $p_T^2/Q^2$ to find

$$F \propto \int dp_T^2 f(\xi, p_T^2) + \int dp_T^2\, \frac{p_T^2}{Q^2}\, \xi\, \frac{\partial}{\partial \xi} f(\xi, p_T^2) + O\left(\frac{\langle p_T^4\rangle}{Q^4}\right). \tag{14}$$

Compute the $\xi$ moments of the structure function $F$:

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = \int_0^1 d\xi\, \xi^n \int dp_T^2 f(\xi, p_T^2) - (n+1)\int_0^1 d\xi\, \xi^n \int dp_T^2\, \frac{p_T^2}{Q^2} f(\xi, p_T^2) + O\left(\frac{\langle p_T^2\rangle^2}{Q^4}\right), \tag{15}$$

where to obtain the second term we have integrated by parts and discarded surface terms. If

$$\frac{\int_0^1 d\xi\, \xi^n \int dp_T^2\, \frac{p_T^2}{Q^2} f(\xi, p_T^2)}{\int_0^1 d\xi\, \xi^n \int dp_T^2 f(\xi, p_T^2)} \simeq \frac{\langle p_T^2\rangle}{Q^2} \tag{16}$$

independent of $n$, Eq. (15) becomes

$$\int_0^1 d\xi\, \xi^n F(\xi, Q^2) = A_n\left[1-(n+1)\frac{\langle p_T^2\rangle}{Q^2}+\cdots\right]. \tag{17}$$

The effect of a nonzero $\langle p_T^2\rangle$ is of the twist-4 form, but note that it even has the $n$ dependence obtained by DGP from QCD.

Keeping successively higher powers of $p_T^2$ in the derivation leading to Eq. (17) will introduce powers of $1/Q^2$ times higher moments of the $p_T$ distribution. These moments of the $p_T$ distribution as a function of $\xi$ are contained in the matrix elements of successively higher twists (just as twist-2 operators give the $\xi$ distribution times $p_T^0$).

Higher twists (>2) also include final-state interactions (with high twist the initial-final distinction breaks down). The most significant final-state effect is due to the exchange of gluons between the struck and spectator quarks. This is an attractive interaction because the final state has the color quantum numbers of the target. We know this attraction has a particularly striking, nonscaling effect for small $W$, i.e., $1 \lesssim W \lesssim 2$ GeV (at high $W$ the colors are presumably shielded by pair production). An attractive final-state interaction increases the cross section; so moments of the structure functions will have the form

$$A_n\left(1+n\frac{\mu^2}{Q^2}+\cdots\right), \tag{18}$$

where the + sign is dictated by the attraction, and the form $n\mu^2/Q^2$ is determined by the phenomenon being fixed in $W$ [see Eq. (1b)]; $\mu$ is some small mass, corresponding to $1 \lesssim W \lesssim 2$ GeV.

The OPE analysis suggests

$$A_n\left(1+\frac{B_{n,1}}{A_n}n\frac{M_0^{\,2}}{Q^2}+\cdots\right). \tag{19}$$

*A priori*, we expect the parton parameters $\langle p_T^{\,2}\rangle$, $\mu^2$ and the QCD parameters $B_{n,1}M_0^{\,2}/A_n$ to all be of order $\Lambda^2$, where $\Lambda$ sets the coupling strength and was determined to be $500 \pm 200$ MeV in DGP from the logarithmic scaling violations in electroproduction [see Eq. (5)]. Indeed, we found that $B_{n,1}M_0^{\,2}/A_n = (350\text{–}400\ \text{MeV})^2$, independent of $n$ over the range $0 \leq n \leq 30$. The observed + sign in Eqs. (6) and (19) suggests that final-state interactions are in fact more important than the initial $p_T \neq 0$ in determining $1/Q^2$ effects.

## V. SOME REMAINING ISSUES

### Nonperturbative effects

Virtually all important *power* violations of scaling cannot be determined by a straightforward expansion in the coupling constant. They are necessarily nonanalytic in the strong-interaction quark-gluon effective coupling constant, vanishing to all orders in the coupling. Since the DGP analysis uses perturbation theory, is it not necessarily incomplete and, hence, inadequate? This is the "specter of nonperturbative effects" raised by GTW.

The answer (briefly, to be expanded below) is that we use a framework which is completely general and incorporates nonanalytic effects. We use perturbation theory where it should be reliable, but make allowance for these other necessary effects. A variety of nonperturbative estimates, based on simple phenomenology and dimensional analysis, are employed in putting the theory into useful form. These estimates are then checked for consistency with experiment and are finally replaced by numerical determinations from the data. Until someone can compute $\nu W_2$ from first principles, the analysis of inclusive lepton-hadron scattering will necessarily be a combination of theory and phenomenology.

A related, subtler question is whether there might not be unanticipated nonanalytic effects in places where we have not allowed for them. For example, the OPE coefficient functions, for which we do use a power-series expansion in the effective coupling, may contain nonscaling terms which vanish to all orders. Of course they may, but, to the sensitivity probed in DGP, they do not. Using electroproduction data, we show that there are no unaccounted for effects of order 1 $\text{GeV}^2/Q^2$ for $1 \leq Q^2 \leq 16$ $\text{GeV}^2$; the observed effects of order ~0.1 $\text{GeV}^2/Q^2$ are well accounted for by our estimates, but the data are not good enough to rule out something else of that size or smaller.

The necessity of "nonperturbative" effects arises as follows. In the limit of vanishing bare quark masses, QCD is formally scale invariant. This invariance is only formal because the interactions induce a scale, $\Lambda$. The relation between $\Lambda$ and the dimensionless coupling constant is nonanalytic, e.g., for a mass $M$ such that $\alpha_s(M)$ (the chromodynamic fine-structure constant renormalized at $M$) is small, $\Lambda \approx M \exp[-2\pi/9\alpha_s(M)]$, which is in fact $M$-independent. All physical mass parameters are proportional to $\Lambda$.

The $\xi$ variable incorporates kinematic effects of all orders in $Q^2/\nu^2$. These translate naturally into $m_{\text{proton}}^{\,2}/Q^2$ effects. But since the exact $\xi$-scaling analysis is just kinematics, one must use the physical value of $m_{\text{proton}}$ (and not zero, as it is to all orders in normal perturbation theory). The OPE provides us with other parameters of dimension mass-squared which do not vanish in the real world. In particular, $B_{n,1}M_0^{\,2}/A_n$ [from Eq. (6)] are the ratios of the target matrix elements of twist-4 operators and the corresponding twist-2 operators. We estimate them to be $O(\Lambda^2)$ *a priori*. Assuming that they are indeed $O(\Lambda^2)$, we measure them to be $(350\text{–}400\ \text{MeV})^2$. Note that it is reasonable that $m_{\text{proton}}^{\,2}$ be much larger than $M_0^{\,2}$ because the *a priori* estimate of $m_{\text{proton}}$ is $3\Lambda$, where 3 is the number of quark colors.

What can we say of possible "nonperturbative" effects besides $m_{\text{proton}}$ and $M_0$? Our method of measuring $M_0$ and its numerical result ensures that there are no large unaccounted for effects. Recall that to measure $M_0$ in DGP we subtracted the twist-2, smooth function $F_s(\xi, Q^2)$ from the actual cross section and looked at moments where the difference was small but statistically significant. For $n/Q^2 \approx 2\ \text{GeV}^{-2}$ and $1 \leq Q^2 \leq 15\ \text{GeV}^2$, the moments drop by two decades, but the fractional difference is stable around 30%. We interpret this as confirmation of the form of Eq. (6) with $B_{n,k}M_0^{\,2}/A_n = (350\text{–}400\ \text{MeV})^2$, at least for $k = 1$. Any remaining scaling violations must be sufficiently weak over this $Q^2$ range so as to be undetectable given the present experimental errors.

### $\xi$ scaling for free fields

Free-field theories (such as the Gross-Neveu model[11] in the limit $N\to\infty$) scale in $x$. In particular, $\nu W_2 \propto \delta(x-1)$, where $x=Q^2/2\mu\nu$ and $\mu$ is the mass of the free field. We wish to point out here that they also scale in $\xi$, even though $\xi\neq x$; so there is no inconsistency or even awkwardness regarding $\xi$ and free fields.

To see that this is the case, remember that for free fields, the constituent and target mass are one and the same. Recall the formula from GP for $\xi$ including the constituent mass,

$$\xi = x\frac{1+(1+4\mu^2/Q^2)^{1/2}}{1+(1+4x^2\mu^2/Q^2)^{1/2}}. \tag{20}$$

For $x=1$, $\xi=1$; so $\delta(x-1)\propto\delta(\xi-1)$, and $\nu W_2 \propto \delta(\xi-1)$ which evidently $\xi$-scales.

### Heavy quarks

Barbieri *et al.*[4] point out that threshold $\theta$ functions, of obvious physical origin, are not always manifest in the OPE analysis, particularly in the case of heavy struck quarks. The resolution to this apparent inadequacy is the fact that only the product of coefficient function times target matrix element has physical significance. The missing $\theta$ functions reside in Mellin transforms of certain matrix elements. These are discussed briefly in GP. But it is easy to draw a qualitative picture in parton language.

The probability for finding a light parton in a given configuration depends only weakly on $\log Q^2$ for $Q^2 \gtrsim 1$ GeV$^2$. However, the probability of "finding" a heavy parton (i.e., on-shell) in the proton depends critically on $Q^2$ and $\nu$ until one is well above the relevant thresholds. In the field-theoretic framework, this can be seen in the strong $n$ and $Q^2$ dependence of anomalous dimensions of heavy-quark operators.[1]

Heavy-quark thresholds will be accompanied by a repeat version of local duality, where resonant final states will dominate the cross section for $W_{\mathrm{threshold}} \leq W \lesssim W_{\mathrm{threshold}} + O(1\ \mathrm{GeV})$, but will average (in the manner of Bloom and Gilman) to a smooth function that scales in the appropriate variable $\xi$.[1]

### $\xi\to 0$

The limit $\xi\to 0$ is no different from the limit $x\to 0$ because $\xi = x(1-x^2m^2/Q^2+\cdots)$ (for massless quarks). There are two difficulties which stand in the way of a study of the region of small $\xi$. Hence we restricted our attention to $\xi \gtrsim \frac{1}{3}$ in DGP. But it may be worth remembering what these difficulties are because any progress in understanding them would allow an extension of the field-theoretic analysis of lepton-hadron scattering.

Somewhere at small $\xi$ resides a non-negligible gluon distribution. Even though lepton currents do not couple to gluons directly, gluons can effect scaling violations through transitions to quark pairs.

The limit $\xi\to 0$ is governed by negative $n$ in the language of moments. Specific calculations are done for integer $n\geq 0$. The approximate analytic continuation to negative $n$ is contingent on the nature of singularities in the complex $n$ or angular momentum plane. Perturbation theory does not give a realistic picture of these singularities.


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